EAN 2014
52nd conference on experimental stress analysis

System Identification for Underdamped Mechanical Systems

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Keywords: Modal analysis, natural frequencies, damping ratio, characteristic equation.

Abstract. There are many ways to model and to analyze discrete event systems. In general these systems lead to a non-linear characteristic equation description in linear algebra. This paper presents an analytical method for solving the characteristic equation of higher order, which arise when solving ordinary differential equations of motion of rigid body systems with $2 \le p° \le 10$ degrees of freedom. The objective of this work was to express the characteristic equation in the form of product quadratic polynomial, from which the modal components could be found. To validate the model, the modal parameters extraction technique – Ibrahim Time Domain (ITD) – was used to extract modal parameters from artificial data developed in MATLAB environment. The extracted modal components were compared to those obtained from the analytical model.

Introduction

The equations of motion are fundamental in investigating the vibration of rigid bodies coupled with elastic and dissipative elements. The basic model of road and railway vehicles and some simple operational machines are presented as examples of rigid bodies system with lower degree of freedom $2 \le p° \le 10$ approximately. The current work presents an analytical method for solving the characteristic equation of the rigid body, which arises when solving ordinary differential equations.

The investigation focuses on the estimation of natural frequencies, damping ratio and the system's mode shapes. Accuracy of the modal components obtained using the proposed method is evaluated by comparing results extracted from artificial experimental data processed in MATLAB environment. In this paper, the Ibrahim time domain is used to extract modal parameters from the artificial data. Ibrahim Time Domain ITD method is a time domain method and it uses the Impulse Response Function (IRF) data to indentify modal parameters. A complete development of this method is found in [1,4]. A vast amount of work along this line has been studied and published by many researchers. The general approaches to modal testing and their reviews can be found in [3] and reference [4] gives a broader explanation regarding modal parameter extraction techniques.

Analytical Model

Assuming a linear coupling of elastic and dissipative elements, and small displacement of the rigid bodies, the equations of motion of different kind of mechanical systems may be obtained in the general form as,

$$\mathbf{M}\ddot{q}(t) + \mathbf{B}\dot{q}(t) + \mathbf{K}q(t) = \mathbf{Q}_j(t), \qquad (1)$$

where **M** – the mass matrix of the system, **B** – the damping matrix, **K** – the stiffness matrix, $\mathbf{Q}_j(t)$ – is the generalized excitation vector, $q(t)$ - vectors of the generalized coordinate.

After dividing Eq. 1 by the respective diagonal element of the mass matrix **M** and after the Laplace transformation for the zero initial conditions $q_0(t)$ and $\dot{q}_0(t)$, the system of differential equations is transformed to the system of algebraic Eq. 2 in the *s*-domain.

$$\mathbf{G}\cdot\bar{q}(s)=\bar{f}(s)\ , \tag{2}$$

where *s* is the parameter of transformation, $\bar{q}(s)$ and $f(s)$ are the vectors of generalized coordinates $q(t)$ and generalized forces $f(t)$. According to [1,2,3], for solving the system of algebraic Eqs. 2, it is possible due to small number of equations (for $p° \leq 10$), to apply the Cramer rule as follows,

$$q_j(s)=\sum_{i=1}^{3}(-1)^{j+i}F_i(s)\frac{D_{ji}(s)}{D(s)}\ . \tag{3}$$

Before the inverse transformation of the generalized coordinate of Eq. 3, it is necessary to determine the poles of polynomial $D(s)$ of Eq. 3. The roots of polynomial $D(s)$ are complex conjugate $s_k=-\mathrm{Re}\,s_k\pm\mathrm{Im}\,s_k$ and their product is a quadratic polynomial for damped mechanical systems based on the assumptions made. According to [2], the poles of the characteristic equation of order $n = 2p$ can be determined as follows:

$$D(s)=\sum_{i=0}^{2n}A_{2(n-i)}p^{2(n-1)}=\prod_{k=1}^{n/2}\left(s^2+p_k\cdot s+r_k\right)\ , \tag{4}$$

where $r_k=(\mathrm{Im}\,s_k)^2$ and $p=2\cdot\mathrm{Re}\,s_k$ for the mechanical system under investigation. The real positive coefficients of polynomial (4) $A_{2(n-i)}$ are the element functions of the mass, damping and stiffness matrices. Modification of the fraction of the denominator of Eq. 3 or simply Eq. 4 yields

$$s^2+p_k\cdot s+r_k=\left(s^2+b_k\right)^2+\omega_k^2\ ,\ \text{for } k=1,2,\ldots,n/2\ , \tag{5}$$

where parameter $\omega_k^2=\omega_{0k}^2-b_k^2$ of the quadratic polynomial is the damped natural frequency, $\omega_{0k}^2=r_k$ is the square inherent circular natural frequency of undamped mechanical systems and $b_k=\frac{p_k}{2}$ is the coefficient of linear viscous damping. The accuracy of results obtained using Eq. 4 is evaluated by comparing results extracted from artificial experimental data processed in MATLAB environment.

Artificial Experimental Data

In order to create artificial experiment data to be used later in this paper, it is necessary to determine original $q_j(t)$ of the corresponding image $\bar{q}_j(s)$, it is also suitable to transform

Eq. 3 to the form of convolution integral. Therefore it is necessary to transfer or modify the ratio in Eq. 9 to the sum of partial fractions in the form

$$\frac{D_{ij}(s)}{D(s)} = \frac{\sum_{r=1}^{n/2}\left[\left(K_{ji,r}\cdot s + L_{ji,r}\right)\cdot \prod_{\substack{k=1\\k\neq i}}^{n/2}\left(s^2 + p_k\cdot s + r_k\right)\right]}{\prod_{k=1}^{n/2}\left(s^2 + p_k\cdot s + r_k\right)} = \sum_{r=1}^{n/2}\frac{K_{ji,r}\cdot s + L_{ji,r}}{s^2 + p_k\cdot s + r_k}\ , \tag{6}$$

where constants $K_{ji,r}$ and $L_{ji,r}$ for j = 1, 2, …, n/2, i = 1, 2, …, n/2, r = 1, 2, …, n/2, can be determined from the condition of the coefficients equality of the identical powers of parameter s in the numerator of fractional Eq. 6. Substituting Eq. 6 into Eq. 3 for the determination of the image of generalized coordinates $\overline{q}(s)$, for j = 1, 2, …, n/2, can be modified as follows

$$\overline{q}_j(s) = \sum_{i=1}^{n/2}(-1)^{j+i}\cdot \overline{f}_i(s)\cdot \sum_{k=1}^{n/2}\frac{K_{ji,k}\cdot s + L_{ji,k}}{s^2 + p_k\cdot s + r_k}\ . \tag{7}$$

After inverse transformation of Eq. 7 for the function of generalized coordinate $q_j(t)$, for j = 1, 2, 3,…, n, the form of the sum of convolution integral is obtained as follows,

$$q_j(t) = \sum_{i=1}^{n}(-1)^{j+i}\sum_{k=1}^{n}\left[K_{ji,k}\int_0^t F_i(\tau)e^{-\beta_k(t-\tau)}\cos\,\omega_k(t-\tau)d\tau + \right.$$
$$\left.\frac{L_{ji,k}-\beta_k K_{ji,k}}{\omega_k}\int_0^t F_i(\tau)e^{-\beta_k(t-\tau)}\sin\,\omega_k(t-\tau)d\tau\right]\ , \tag{8}$$

where j-th component of vector of generalized coordinates, $q_j(t)$ is the sum integrals convolution multiplied by the i-th generalized kinematic excitation elements, $F_j(t)$ designated by the product of spring constant and height of the road or rail surface unevenness (for road and rail vehicle models) and by product of damping coefficient b_{jik}, and time derivative of height contact place of m-index wheel at specific crossing velocity, to the k-th harmonic component with its own natural frequency ω_k. K_{jik} and L_{jik} are unknown coefficients of amplitude, depending on the mechanical properties of the system under consideration. Vector components of the kinematic excitation function are given in the range $0 \leq t$. Eq. 8 shows the solution for a linear viscous damped mechanical system. Using Eq. 8 the MATLAB code was implemented to produce so-called artificial experimental data.

Ibrahim Time Domain Method

The ITD method uses the IRF data to identify modal parameters. It makes an eigenvalue problem from the IRF data and solves the problem to derive the natural frequencies, damping ratio and the system’s mode shapes. References [1] and [4] give a complete derivative of this method. The ITD uses the free response of the tested system. According to [1] the approach is to sample the same response data, as in Eq. 8, but with a duration τ shift.

For the i-th measurement this means [1]:

$$q_i\left(t+\tau\right)=\sum_{r=1}^{2N}\varphi_{ir}e\lambda_r^{(t+\tau)}\ . \tag{9}$$

For all the response points, a vector form of Eq. 9 can be written as:

$$\left\{Y_j\right\}=\left\{q\left(t_j+\tau\right)\right\}=\left[\{\varphi\}_1 e^{\lambda_1\tau},\{\varphi\}_2 e^{\lambda_2\tau},\cdots\{\varphi\}_{2N} e^{\lambda_{2N}\tau}\right]\begin{Bmatrix} e^{\lambda_1 \tau j} \\ e^{\lambda_2 \tau j} \\ \cdots \\ e^{\lambda_{2N} \tau j}\end{Bmatrix}\ . \tag{10}$$

From this analysis, three sampled data matrices [**X**], [**Y**], [**Z**] and three modal vector matrices [**φ**], [P], [Q] are created. Next step is to assemble an eigenvalue problem from these matrices, from which the complex natural frequencies and mode shapes can be derived.

$$[\mathbf{W}][\mathbf{V}]^{-1}[\boldsymbol{\mu}]=[\boldsymbol{\mu}][\boldsymbol{\Lambda}]\ . \tag{11}$$

The upper half of $[\boldsymbol{\mu}]$, is the complex mode shape matrix. Matrix $[\boldsymbol{\Lambda}]$ contains the complex eigenvalues [1,4]. These eigenvalues can be converted to the natural frequencies and damping ratio of the tested mechanical system. The *r*-th eigenvalue is assumed to be in the form

$$\boldsymbol{\Lambda}_r = e^{\lambda_r \tau} = \alpha_r + j\beta_r\ . \tag{12}$$

The eigenvalues, denoted as in Eq. 12, are the *r*-th complex natural frequency

$$\lambda_r = -n_r + j\omega_{kr}\ . \tag{13}$$

Combining Eq. 11 with Eq. 12 yields

$$n_r = -\frac{1}{2\tau}\ln(\alpha_r^2+\beta_r^2)\ , \tag{14}$$

$$\omega_{kr} = \frac{1}{\tau}\tan^{-1}\frac{\beta_r}{\alpha_r}\ . \tag{15}$$

From Eqs. 16 and 17, the undamped natural frequencies and damping ratio can be found easily,

$$\omega_{0kr} = \sqrt{\omega_{kr}^2 + n_r^2}\ , \tag{16}$$

$$\zeta_r = \frac{n_r}{\omega_{0kr}}\ . \tag{17}$$

Results and Conclusion

Fig. 1 shows, a three-degree of freedom theoretical model considered in this paper to obtain the system's dynamic properties (natural frequencies, damped natural frequencies, damping ratio and mode shapes). The objective of the model is to investigate the effects of various eccentricities due to eccentric weight loading. This investigation is not the subject of this paper. Artificial data were developed in the MATLAB based on the model and the ITD was used to extract modal components from these data. The input parameters to referring to the MATLAB code are provided in Table 1.

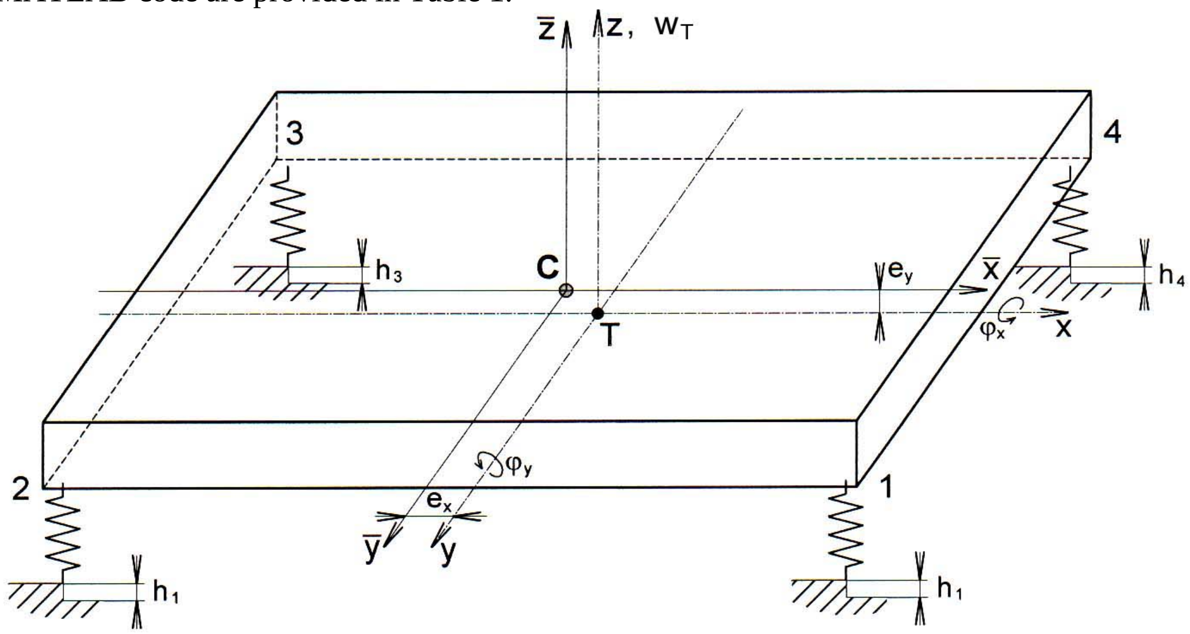

Fig. 1. Theoretical model.

Table 1. Model parameters.

K	16.49 ± 0.12 N·mm^{-1}	m	43.702 kg
L_x	0.690 m (690 mm)	I_x	288582.52 kg·mm^2
L_y	0.280 m (280 mm)	I_y	1736939.6 kg·mm^2
L_z	0.029 m (29 mm)	I_z	2019396.6 kg·mm^2

Table 2 and Table 3 show the results obtained by the proposed analytical method, and also the results obtained from the artificial data via the ITD extraction technique are shown. The differences between the analytical model and the ITD were found to be in the range from 0.38 % to 9.34 % for the natural frequency, and from 4.20 % to 9.66 % for the damping ratio.

Table 2. Analytical and estimated modal components.

Mode	**Analytical model**		**ITD**		**Error [%]**
1	N. freq. [Hz]	3.84870	N. freq. [Hz]	3.86346	0.38
	Damp. ratio	0.08457	Damp. ratio	0.08828	4.20
2	N. freq. [Hz]	7.11782	N. freq. [Hz]	6.51007	9.34
	Damp. ratio	0.02581	Damp. ratio	0.02857	9.66
3	N. freq. [Hz]	9.87793	N. freq. [Hz]	9.45718	4.44

	Damp. ratio	0.013814	Damp. ratio	0.01464	5.64

Table 3. System's mode shapes.

Mode 1	**Mode 2**	**Mode 3**
Analytical model		
$1.31484 \cdot 10^{-13}$	$-4.16157 \cdot 10^{-17}$	$4.78045 \cdot 10^{-14}$
$-1.97612 \cdot 10^{-11}$	$4.25042 \cdot 10^{-15}$	$7.44883 \cdot 10^{-12}$
$2.58578 \cdot 10^{-9}$	$1.282329 \cdot 10^{-9}$	$1.73104 \cdot 10^{-6}$
ITD		
$1.53122 \cdot 10^{-12}$	$-5.39776 \cdot 10^{-13}$	$4.25045 \cdot 10^{-15}$
$-3.33551 \cdot 10^{-12}$	$2.22345 \cdot 10^{-9}$	$1.28233 \cdot 10^{-9}$
$4.77356 \cdot 10^{-8}$	$1.57340 \cdot 10^{-8}$	$1.27328 \cdot 10^{-6}$

On the basis of the results obtained it can be confirmed that an acceptable level of agreement can be achieved between "test data" and the results obtained by the proposed method.

Conclusion

Obtaining accurate modal parameters that can be used for validating analytical models is a non-trivial and exacting task. This paper presented an analytical method for solving the characteristic equation of higher order, which arise when solving ordinary differential equations of motion of rigid body systems with $2 \le p° \le 10$ degrees of freedom, from which the system's modal components can be found easily. To validate the model, artificial data were reproduced in MATLAB, and the results compared. The results confirmed an acceptable level of agreement between the analytical model proposed and the modal technique – ITD.

Acknowledgement

The research work is supported by the Student Grant Agency (SGA) – UJEP, Czech Republic.

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